ON NONEMPTINESS OF NEWTON STRATA IN THE B_{dR}^{+}-GRASSMANNIAN FOR GSp_{2n}

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ABSTRACT. We build upon our previous work [Hon22] to study the Newton stratification on the B_{dR}^{+}-Grassmannian for GSp_{2n}. Our main result gives an explicit classification of all nonempty Newton strata associated to a Frobenius-conjugacy class satisfying certain conditions on the Newton polygon.

CONTENTS

1. INTRODUCTION

The B_{dR}^{+}-Grassmannian is a p-adic analogue of the complex affine Grassmannian. It has played a fundamental role in a number of major developments in p-adic geometry, including the work of Caraiani-Scholze [CS17] on the cohomology of unitary Shimura varieties, the work of Scholze-Weinstein [SW20] on the general construction of local Shimura varieties, and the work of Fargues-Scholze [FS21] on the geometrization of the local Langlands correspondence. It has also been used as a central tool for studying the p-adic period domain by many authors, such as Chen-Fargues-Shen [CFS21], Shen [She19], Chen [Che20], Viehmann [Vie21], Nguyen-Viehmann [NV21], and Chen-Tong [CT22].

In order to describe the geometry of the B_{dR}^{+}-Grassmannian, we consider a natural stratification called the *Newton stratification*. Let G be a connected reductive group over a finite extension E of $\mathbb{Q}_p$. We denote by Gr_G the B_{dR}^{+}-Grassmannian associated to G, and by $\mathrm{Gr}_{G,\mu}$ the Schubert cell associated to a dominant cocharacter μ of G. In addition, we write $X = X_E$ for the Fargues-Fontaine curve over E and Bun_G for the stack of G-bundles on X. The work of Fargues [Far20] gives a natural bijection between the topological space $|\mathrm{Bun}_G|$ of Bun_G and the set $B(G)$ of Frobenius-conjugacy classes of $G(\breve{E})$, where $\breve{E}$ denotes the p-adic completion of the maximal unramified extension of E. If we fix an element $b \in B(G)$ and a complete algebraically closed extension C of E, we get a natural map

$$\mathrm{Newt}_b : \mathrm{Gr}_G(C) \longrightarrow |\mathrm{Bun}_G| \simeq B(G)$$

induced by the theorem of Beauville-Laszlo [BL95]. For each Schubert cell $\mathrm{Gr}_{G,\mu}$, the Newton stratification associated to b is a decomposition

$$\mathrm{Gr}_{G,\mu} = \bigsqcup_{b' \in B(G)} \mathrm{Gr}_{G,\mu,b}^{b'}$$

where $\mathrm{Gr}_{G,\mu,b}^{b'}(C)$ is the preimage of b' in $\mathrm{Gr}_{G,\mu}(C)$ under the map Newt_b.

In this paper, we study the Newton stratification of minuscule Schubert cells in the B_{dR}^+-Grassmannian for GSp_{2n}. For a precise statement of our main result, let us recall some basic facts regarding the group GSp_{2n}. As observed by Kottwitz [Kot85], an element in $B(\mathrm{GSp}_{2n})$ is determined by an invariant called the *Newton polygon*, which is a concave polygon on the interval $[0, 2n]$ with integer breakpoints. In addition, we may represent each dominant cocharacter of GSp_{2n} as a $2n$-tuple of descending integers, or equivalently as a concave polygon on the interval $[0, 2n]$ where the slope on $[i-1, i]$ is given by the i-th entry of the $2n$-tuple. After multiplication by a central cocharacter, a minuscule dominant cocharacter of GSp_{2n} is either trivial or represented by the tuple $(\underline{1}^{(n)}, \underline{0}^{(n)})$, where we write $\underline{d}^{(n)} := (d, \cdots, d) \in \mathbb{Z}^n$ for each $d \in \mathbb{Z}$. Since multiplication by a central cocharacter induces a natural bijection between Newton strata on Schubert cells (as noted in Proposition 5.5), it suffices to consider the minuscule cocharacter represented by $(1^{(n)}, 0^{(n)})$.

Theorem 1.1. *Let b and b' be elements in $B(\mathrm{GSp}_{2n})$ with their Newton polygons respectively denoted by $\nu(b)$ and $\nu(b')$. Assume that the difference between any two distinct slopes in $\nu(b)$ is greater than 1. For the cocharacter μ of GSp_{2n} represented by the tuple $(\underline{1}^{(n)}, \underline{0}^{(n)})$, the Newton stratum $\mathrm{Gr}_{\mathrm{GSp}_{2n},\mu,b}^{b'}$ is nonempty if and only if the following conditions are satisfied:*

(i) *The polygon $\nu(b')$ lies below the polygon $\nu(b) + \mu^*$ with the same endpoints, where μ^* denotes the cocharacter of GSp_{2n} represented by the tuple $(\underline{0}^{(n)}, \underline{-1}^{(n)})$.*

(ii) *For $i = 1, 2, \cdots, 2n$, we have an inequality*

$$\lambda_i(\nu(b')) \leq \lambda_i(\nu(b)) \leq \lambda_i(\nu(b')) + 1$$

where $\lambda_i(\nu(b))$ and $\lambda_i(\nu(b'))$ respectively denote the slopes of $\nu(b)$ and $\nu(b')$ on $[i-1, i]$.

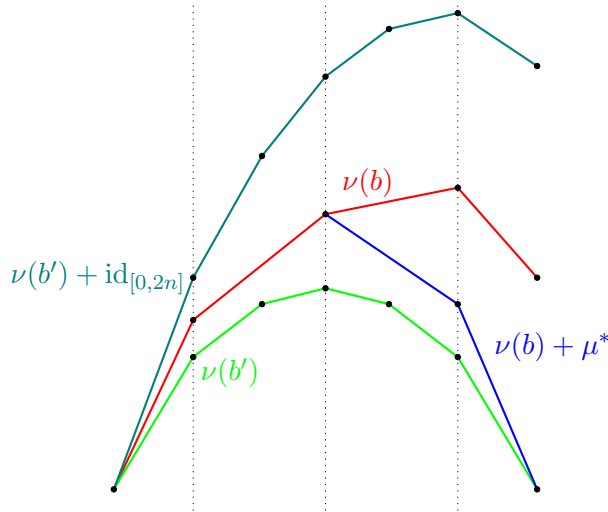(iii) *For each breakpoint of $\nu(b)$, there exists a breakpoint of $\nu(b')$ with the same x-coordinate.*

FIGURE 1. Illustration of the conditions in Theorem 1.1

Let us briefly sketch our proof of Theorem 1.1. We consider the natural embedding $\mathrm{GSp}_{2n} \hookrightarrow \mathrm{GL}_{2n}$ and identify $b, b' \in B(\mathrm{GSp}_{2n})$ with their images under the induced map $B(\mathrm{GSp}_{2n}) \longrightarrow B(\mathrm{GL}_{2n})$. Then we obtain a natural map

$$\mathrm{Gr}_{\mathrm{GSp}_{2n}, \mu, b}^{b'} \longrightarrow \mathrm{Gr}_{\mathrm{GL}_{2n}, \mu, b}^{b'},$$

and consequently deduce the necessity part of Theorem 1.1 from a previous result of the author [Hon22]. For the sufficiency part of Theorem 1.1, we regard the Newton polygons $\nu(b)$ and $\nu(b')$ as rational cocharacters of GSp_{2n}. Let M denote the centralizer of $\nu(b)$, which is a Levi subgroup of GSp_{2n}. By construction, the element $b \in B(\mathrm{GSp}_{2n})$ admits a *reduction* $b_M \in B(M)$; in other words, b is the image of b_M under the natural map $B(M) \longrightarrow B(\mathrm{GSp}_{2n})$. Moreover, the condition (iii) implies that $b' \in B(\mathrm{GSp}_{2n})$ admits a reduction $b'_M \in B(M)$. Hence we obtain a natural map

$$\mathrm{Gr}_{M, \mu, b_M}^{b'_M} \longrightarrow \mathrm{Gr}_{\mathrm{GSp}_{2n}, \mu, b}^{b'}.$$

Let us identify the Newton polygons of b_M and b'_M respectively with $\nu(b)$ and $\nu(b')$. Since the Newton polygon of b_M is central in M, we deduce the nonemptiness of $\mathrm{Gr}_{M, \mu, b_M}^{b'_M}$ from the conditions in Theorem 1.1 by the work of Rapoport [Rap18], Chen-Fargues-Shen [CFS21], Shen [She19], and Viehmann [Vie21], thereby establishing the nonemptiness of $\mathrm{Gr}_{\mathrm{GSp}_{2n}, \mu, b}^{b'}$.

We expect that our strategy can be used to establish an analogous statement of Theorem 1.1 for other classical groups. We plan to update this manuscript as we make progress on other groups.

2. Notations and conventions

Throughout the paper, we consider the following data:

- E is a finite extension of $\mathbb{Q}_p$ with uniformizer π.
- C is a complete and algebraically closed extension of E.
- G is a reductive group over E with Borel subgroup B and maximal torus $T \subseteq B$.

Given such data, we set up the following notations:

- Γ is the absolute Galois group of E.
- $\breve{E}$ is the p-adic completion of the maximal unramified extension of E.
- σ is the Frobenius automorphism of $\breve{E}$
- $B(G)$ is the set of σ-conjugacy classes of $G(\breve{E})$.
- $(X^*(T), \Phi, X_*(T), \Phi^\vee)$ is the absolute root datum of G.
- Ω is the Weyl group of G.

In addition, we use the following general notations:

- Given a valued field K, we write $\mathcal{O}_K$ for its valuation ring.
- Given a perfectoid ring R, we write $R^\flat$ for its tilt.
- Given a perfect $\mathbb{F}_p$-algebra A, we write $W(A)$ for the ring of Witt vectors over A.
- Given an abelian group M equipped with an action of a group H, we write M^H and M_H respectively for the subgroups of H-invariants and H-coinvariants.

3. G-BUNDLES ON THE FARGUES-FONTAINE CURVE

Definition 3.1. Let us fix a pseudouniformizer ϖ of $C^\flat$. In addition, we write q for the number of elements in the residue field of E and set $W_{\mathcal{O}_E}(\mathcal{O}_{C^\flat}) := W(\mathcal{O}_{C^\flat}) \otimes_{W(\mathbb{F}_q)} \mathcal{O}_E$.

(1) We define the *perfectoid punctured unit disk* associated to the pair $(E, C^\flat)$ by

$$\mathcal{Y} := \mathrm{Spa}(W_{\mathcal{O}_E}(\mathcal{O}_{C^\flat}) \setminus \{|\pi[\varpi]| = 0\}$$

where $[\varpi]$ denotes the Teichmuller lift of ϖ in $W_{\mathcal{O}_E}(\mathcal{O}_{C^\flat})$.

(2) We write ϕ for the automorphism of $\mathcal{Y}$ induced by the q-Frobenius automorphism on $W_{\mathcal{O}_E}(\mathcal{O}_{C^\flat})$, and define the *adic Fargues-Fontaine curve* associated to the pair $(E, C^\flat)$ by

$$\mathcal{X} := \mathcal{Y}/\phi^{\mathbb{Z}}.$$

(3) We define the *schematic Fargues-Fontaine curve* associated to the pair $(E, C^\flat)$ by

$$X := \mathrm{Proj}\left(\bigoplus_{n \geq 0} B^{\phi = \pi^n}\right)$$

where B denotes the ring of global sections on $\mathcal{Y}$.

Remark. The construction of the adic Fargues-Fontaine curve relies on the fact that the action of ϕ on $\mathcal{Y}$ is properly discontinuous.

Theorem 3.2 ([Ked16, Theorem 4.10], [FF18, Théorème 6.5.2], [KL15, Theorem 8.7.7]). *We have the following facts regarding $\mathcal{X}$ and X:*

(1) $\mathcal{X}$ is a Noetherian adic space over E.

(2) X is a Dedekind scheme over E.

(3) There exists an equivalence of the categories of vector bundles on $\mathcal{X}$ and X, induced by a natural map $\mathcal{X} \longrightarrow X$ of locally ringed spaces.

Remark. The statement (3) in Theorem 3.2 allows us to identify G-bundles on $\mathcal{X}$ with G-bundles on X.

Definition 3.3. Given an element $b \in B(G)$, we define the associated G-bundle $\mathcal{E}_b$ on $\mathcal{X}$ (or on X) to be the descent along the map $\mathcal{Y} \longrightarrow \mathcal{Y}/\phi^{\mathbb{Z}} = \mathcal{X}$ of the trial G-bundle on $\mathcal{Y}$ equipped with the ϕ-linear automorphism induced by b.

Theorem 3.4 ([Far20, Théorème 5.1]). *There is a natural bijection*

$$B(G) \xrightarrow{\sim} H^1_{\acute{e}t}(\mathcal{X}, G)$$

which maps each $b \in B(G)$ to the isomorphism class of $\mathcal{E}_b$.

Definition 3.5. We define the *Newton set* of G by

$$\mathcal{N}(G) := (X_*(T)_{\mathbb{Q}}/\Omega)^{\Gamma},$$

where we write $X_*(T)_{\mathbb{Q}} := X_*(T) \otimes_{\mathbb{Z}} \mathbb{Q}$ for the space of rational cocharacters of G, and the *algebraic fundamental group* of G by

$$\pi_1(G) := X_*(T)/\langle \Phi^\vee \rangle,$$

where $\langle \Phi^\vee \rangle$ denotes the subgroup of $X_*(T)$ generated by $\Phi^\vee$.

Proposition 3.6 ([Kot85, §4], [Kot97, §6]). *There exist unique natural transformations*

$$\nu : B(\cdot) \longrightarrow \mathcal{N}(\cdot) \quad and \quad \kappa : B(\cdot) \longrightarrow \pi_1(\cdot)_\Gamma$$

of set-valued functors on the category of connected reductive groups over E such that the following statements hold:

(1) *Under the map $\nu : B(G) \longrightarrow \mathcal{N}(G)$, the image of $b \in B(G)$ with a representative $\tilde{b}$ is given by a unique rational cocharacter $\nu_{\tilde{b}}$ of G for which there exist an integer $m > 0$ and an element $c \in G(\breve{E})$ with the following properties:*

   (i) *$m\nu_{\tilde{b}}$ lies in $X_*(T)$.*

   (ii) *$\mathrm{Int}(c) \circ m\nu_{\tilde{b}}$ is defined over the fixed field of σ^m in $\breve{E}$, where $\mathrm{Int}(c)$ denotes the inner automorphism mapping each $g \in G(\breve{E})$ to cgc^{-1}.*

   (iii) *We have $c \cdot \tilde{b} \cdot \sigma(\tilde{b}) \cdots \sigma^m(\tilde{b}) \cdot \sigma^m(c)^{-1} = c \cdot (m\nu_{\tilde{b}})(\pi) \cdot c^{-1}$.*

(2) *We have a commutative diagram*

$$\begin{array}{ccc} \breve{E}^\times & \xrightarrow{\mathrm{val}} & \mathbb{Z} \\ \downarrow & & \| \\ B(\mathbb{G}_\mathrm{m}) & \xrightarrow{\kappa} & \pi_1(\mathbb{G}_\mathrm{m})_\Gamma \end{array}$$

   where $\mathrm{val} : \breve{E}^\times \longrightarrow \mathbb{Z}$ is the π-adic valuation map on $\breve{E}$.

Remark. If G is unramified, there is an explicit description of the map $\kappa : B(G) \longrightarrow \pi_1(G)_\Gamma$ given by Rapoport-Viehmann [RV14, Remark 2.3].

Definition 3.7. We refer to the maps $\nu : B(G) \longrightarrow \mathcal{N}(G)$ and $\kappa : B(G) \longrightarrow \pi_1(G)_\Gamma$ from Proposition 3.6 respectively as the *Newton map* and the *Kottwitz map* of G.

Remark. In line with Theorem 3.4, Fargues [Far20, Propositions 6.6 and 8.1] provides a geometric description of ν and κ in terms of G-bundles on the Fargues-Fontaine curve. We will briefly sketch this description for $G = \mathrm{GL}_n$ after Example 3.13.

Proposition 3.8 ([Kot97, §4.13]). *Every element of $B(G)$ is uniquely determined by its image under the Newton map and the Kottwitz map. In other words, the map*

$$\nu \times \kappa : B(G) \longrightarrow \mathcal{N}(G) \times \pi_1(G)_\Gamma$$

is injective.

Proposition 3.9 ([RR96, Theorem 1.15], [RV14, §2]). *The maps ν and κ fit into a natural commutative diagram*

$$\begin{array}{ccc} B(G) & \xrightarrow{\kappa} & \pi_1(G)_\Gamma \\ \nu \downarrow & & \downarrow \\ \mathcal{N}(G) & \hookrightarrow X_*(T)_\mathbb{Q} \twoheadrightarrow & \pi_1(G)_\Gamma \otimes_\mathbb{Z} \mathbb{Q} \end{array}$$

where the inclusion $\mathcal{N}(G) \hookrightarrow X_*(T)_\mathbb{Q}$ is obtained by identifying $X_*(T)_\mathbb{Q}/\Omega$ with the set of dominant rational cocharacters.

Corollary 3.10. *If $\pi_1(G)_\Gamma$ is torsion free, the Newton map of G is injective.*

Proof. When $\pi_1(G)_\Gamma$ is torsion free, the natural map $\pi_1(G)_\Gamma \longrightarrow \pi_1(G)_\Gamma \otimes_\mathbb{Z} \mathbb{Q}$ is injective. Hence the assertion follows from Propositions 3.8 and 3.9. $\square$

Definition 3.11. We say that an element $b \in B(G)$ is *basic* if $\nu(b)$ is represented by a central rational cocharacter.

Proposition 3.12. *Let $B(G)_{\mathrm{basic}}$ denote the set of basic elements in $B(G)$. The Kottwitz map induces a bijection*

$$B(G)_{\mathrm{basic}} \simeq \pi_1(G)_\Gamma.$$

Example 3.13. Let us discuss the Newton map and the Kottwitz map for some groups.

(1) For GL_n, we have natural identifications

$$\mathcal{N}(\mathrm{GL}_n) \cong \{(\lambda_1, \cdots, \lambda_n) \in \mathbb{Q}^n : \lambda_1 \geq \cdots \geq \lambda_n\} \quad \text{and} \quad \pi_1(\mathrm{GL}_n)_\Gamma \cong \mathbb{Z}.$$

We regard elements of $\mathcal{N}(\mathrm{GL}_n)$ as concave polygons on the interval $[0, n]$ with rational slopes. Then we have the following facts:

(a) The image of the Newton map for GL_n consists of all concave polygons on $[0, n]$ with integer breakpoints, essentially by the classical result of Manin [Man63].

(b) An element $b \in B(\mathrm{GL}_n)$ is basic if and only if its image under the Newton map is a line segment.

(c) The Newton map of GL_n is injective by Corollary 3.10.

(d) Proposition 3.9 yields a commutative diagram

$$
\begin{array}{ccc}
B(\mathrm{GL}_n) & \xrightarrow{\;\;\kappa\;\;} & \pi_1(\mathrm{GL}_n)_\Gamma \cong \mathbb{Z} \\
{\scriptstyle \nu}\downarrow & & \uparrow \\
\mathcal{N}(\mathrm{GL}_n) & \xrightarrow{\;\;\Sigma\;\;} & \mathbb{Q}
\end{array}
$$

where $\sum$ maps each element in $\mathcal{N}(\mathrm{GL}_n)$ to the sum of its entries.

(2) For GSp_{2n}, we consider the natural embedding into GL_{2n} and obtain an identification

$$\mathcal{N}(\mathrm{GSp}_{2n}) \cong \{(\lambda_1, \cdots, \lambda_{2n}) \in \mathcal{N}(\mathrm{GL}_{2n}) : \lambda_i + \lambda_{2n+1-i} \text{ constant}\}.$$

Moreover, we have $\pi_1(\mathrm{GSp}_{2n})_\Gamma \cong \mathbb{Z}$. Then we find the following facts:

(a) An element $(\lambda_1, \cdots, \lambda_{2n}) \in \mathcal{N}(\mathrm{GSp}_{2n})$ lies in the image of the Newton map if and only if it has integer breakpoints with $\lambda_i + \lambda_{2n+1-i} \in \mathbb{Z}$ for $i = 1, \cdots, n$.

(b) An element $b \in B(\mathrm{GSp}_{2n})$ is basic if and only if its image under the Newton map is a line segment.

(c) The Newton map of GSp_{2n} is injective by Corollary 3.10.

(d) Proposition 3.9 yields a commutative diagram

$$
\begin{array}{ccc}
B(\mathrm{GSp}_{2n}) & \xrightarrow{\;\;\kappa\;\;} & \pi_1(\mathrm{GSp}_{2n})_\Gamma \cong \mathbb{Z} \\
{\scriptstyle \nu}\downarrow & & \uparrow \\
\mathcal{N}(\mathrm{GSp}_{2n}) & \xrightarrow{\;\frac{1}{n}\Sigma\;} & \mathbb{Q}
\end{array}
$$

where $\sum$ maps each element in $\mathcal{N}(\mathrm{GSp}_{2n})$ to the sum of its entries.

Remark. Let us provide an explicit description of the Newton map and the Kottwitz map for GL_n in terms of vector bundles on X. By Theorem 3.2, the Harder-Narasimhan formalism applies to the category of vector bundles on X. Under the identifications for $\mathcal{N}(\mathrm{GL}_n)$ and $\pi_1(\mathrm{GL}_n)_\Gamma$ in Example 3.13, $\nu(b)$ and $\kappa(b)$ for each $b \in B(\mathrm{GL}_n)$ are respectively given by the Harder-Narasimhan polygon and degree of $\mathcal{E}_b$. In addition, an element $b \in B(\mathrm{GL}_n)$ is basic if and only if $\mathcal{E}_b$ is semistable. The injectivity of the Newton map for GL_n means that vector bundles on X are determined by its Harder-Narasimhan polygon.

4. The Newton stratification of the B_{dR}^+-Grassmannian

Proposition 4.1 ([Fon82, Proposition 2.4], [KL15, Lemma 3.6.3]). *Every perfectoid affinoid algebra (R, R^+) over E induces a natural surjective homomorphism $\theta_{R,R^+} : W(R^{+\flat}) \twoheadrightarrow R^+$ whose kernel is a principal ideal of $W(R^{+\flat})$.*

Definition 4.2. Let (R, R^+) be a perfectoid affinoid algebra over E.

(1) We refer to the map θ_{R,R^+} in Proposition 4.1 as the *Fontaine map* for (R, R^+)

(2) We define the *positive de Rham ring* associated to R, denoted by $B_{\mathrm{dR}}^+(R)$, to be the completion of $W(R^{+\flat})[1/p]$ with respect to the kernel of θ_{R,R^+}.

(3) We fix an element $t_{R,R^+} \in W(R^{+\flat})$ which generates the kernel of θ_{R,R^+}, and define the *de Rham ring* associated to R by $B_{\mathrm{dR}}(R) := B_{\mathrm{dR}}^+(R)[1/t_{R,R^+}]$.

(4) For $R = C$, we refer to $B_{\mathrm{dR}} := B_{\mathrm{dR}}(C)$ and $B_{\mathrm{dR}}^+ := B_{\mathrm{dR}}^+(C)$ respectively as the *de Rham period ring* and the *positive de Rham period ring*, and write $t := t_{C,\mathcal{O}_C}$.

Remark. As the notation suggests, the construction of $B_{\mathrm{dR}}^+(R)$ and $B_{\mathrm{dR}}(R)$ does not depend on R^+ or t_{R,R^+}.

Proposition 4.3 ([Fon82, Proposition 2.17]). *The ring B_{dR} is a discretely valued field with valuation ring B_{dR}^+ and residue field C.*

Definition 4.4. We define the B_{dR}^+-*Grassmannian* as the functor Gr_G which associates to each perfectoid affinoid E-algebra (R, R^+) the set of pairs $(\mathcal{E}, \beta)$ consisting of a G-torsor $\mathcal{E}$ on $\mathrm{Spec}\,(B_{\mathrm{dR}}^+(R))$ and a trivialization β of $\mathcal{E}$ over $\mathrm{Spec}\,(B_{\mathrm{dR}}(R))$. For simplicity, we write $\mathrm{Gr}_G(C) := \mathrm{Gr}_G(C, \mathcal{O}_C)$.

Proposition 4.5 ([SW20, Proposition 19.1.2]). *There exists a canonical identification*
$$\mathrm{Gr}_G(C) \cong G(B_{\mathrm{dR}})/G(B_{\mathrm{dR}}^+).$$

Proposition 4.6 ([SW20, Corollary 19.3.4]). *Let μ be a dominant cocharacter of G. There exists a locally spatial diamond $\mathrm{Gr}_{G,\mu}$ with*
$$\mathrm{Gr}_{G,\mu}(C) = G(B_{\mathrm{dR}}^+)\mu(t)^{-1}G(B_{\mathrm{dR}}^+)/G(B_{\mathrm{dR}}^+).$$

Remark. We will not use the language of diamonds in a significant way as we will be mainly interested in the C-valued points of Gr_G and $\mathrm{Gr}_{G,\mu}$.

Definition 4.7. Given a dominant cocharacter μ of G, we refer to the locally spatial diamond $\mathrm{Gr}_{G,\mu}$ given by Proposition 4.6 as the *Schubert cell* of Gr_G associated to μ.

Proposition 4.8 ([FF18, Théorèmes 6.5.2 and 7.3.3]). *There exists a closed point ∞ on X satisfying the following properties:*

(i) *$X - \infty$ is isomorphic to $\mathrm{Spec}\,(B_e)$ for some principal ideal domain $B_e \subseteq B_{\mathrm{dR}}$.*

(ii) *The completed local ring at ∞ is canonically isomorphic to B_{dR}^+.*

(iii) *Every G-bundle on X pulls back to a trivial G-bundle via the map $\mathrm{Spec}\,(B_{\mathrm{dR}}^+) \to X$ induced by ∞.*

Definition 4.9. We fix a closed point ∞ on X given by Proposition 4.8, and refer to ∞ as the *distinguished closed point* on X.

Remark. Our terminology is justified by the fact that there is a canonical choice of ∞. The key technical fact is that closed points on X are naturally in bijection with the equivalence classes of untilts of $C^\flat$, as shown by Kedlaya-Liu [KL15, Theorem 8.7.7]. Our canonical choice of ∞ corresponds to the identity map on $C^\flat$.

Proposition 4.10. *The set $H^1_{\text{ét}}(X, G)$ is canonically in bijection with the set of isomorphism classes of triples $(\widetilde{\mathcal{E}}, \widehat{\mathcal{E}}, \beta)$ where*

- $\widetilde{\mathcal{E}}$ *is a G-bundle on $X - \infty$,*
- $\widehat{\mathcal{E}}$ *is a trivial G-bundle on $\text{Spec}(B^+_{\text{dR}})$, and*
- β *is a gluing map of $\widetilde{\mathcal{E}}$ and $\widehat{\mathcal{E}}$ over $\text{Spec}(B_{\text{dR}})$.*

Proof. The assertion is evident by the theorem of Beauville-Laszlo [BL95]. $\qquad\square$

Definition 4.11. Let $\mathcal{E}$ be a G-bundle on X. By a *modification* of $\mathcal{E}$ at ∞, we mean a G-bundle $\mathcal{E}'$ on X together with an isomorphism between the restrictions of $\mathcal{E}$ and $\mathcal{E}'$ on $X - \infty$.

Example 4.12. Let us consider an element $b \in B(G)$ and a point $x \in \text{Gr}_G(C)$. We have $x = gG(B^+_{\text{dR}})$ for some $g \in G(B_{\text{dR}})$ under the identification $\text{Gr}_G(C) \cong G(B_{\text{dR}})/G(B^+_{\text{dR}})$ given by Proposition 4.5. In light of Proposition 4.10 we take a triple $(\widetilde{\mathcal{E}}, \widehat{\mathcal{E}}, \beta)$ corresponding to $\mathcal{E}_b$ and a G-bundle $\mathcal{E}_{b,x}$ on X corresponding to $(\widetilde{\mathcal{E}}, \widehat{\mathcal{E}}, g\beta)$. By construction, $\mathcal{E}_{b,x}$ is a modification of $\mathcal{E}_b$ at ∞.

Remark. It is not hard to see that the isomorphism type of $\mathcal{E}_{b,x}$ does not depend on the choice of $g \in G(B_{\text{dR}})$.

Definition 4.13. Consider an element $b \in B(G)$ and a dominant cocharacter μ of G.

(1) For each $x \in \text{Gr}_G(C)$, we refer to the G-bundle $\mathcal{E}_{b,x}$ in Example 4.12 as the *modification of $\mathcal{E}_b$ at ∞ induced by x.*

(2) For each $b' \in B(G)$, we define the associated *Newton stratum with respect to b* in $\text{Gr}_{G,\mu}$ as the subdiamond $\text{Gr}^{b'}_{G,\mu,b}$ of $\text{Gr}_{G,\mu}$ with

$$\text{Gr}^{b'}_{G,\mu,b}(C) = \{x \in \text{Gr}_{G,\mu}(C) : \mathcal{E}_{b,x} \simeq \mathcal{E}_{b'}\}.$$

Remark. The subdiamond $\text{Gr}^{b'}_{G,\mu,b}$ of $\text{Gr}_{G,\mu}$ is uniquely determined by its set of C-points since $\text{Gr}_{G,\mu}$ is a locally spatial diamond.

Proposition 4.14. *Let μ be a dominant cocharacter of G. For elements $b, b' \in B(G)$, the Newton stratum $\text{Gr}^{b'}_{G,\mu,b}$ is not empty if and only if it contains a C-point.*

Proof. The assertion is evident by definition. $\qquad\square$

Proposition 4.15. *Let $f : G \hookrightarrow \widetilde{G}$ be an embedding of reductive groups over E. We take elements $b, b' \in B(G)$ and write $\widetilde{b}, \widetilde{b}'$ for their images under the map $B(G) \longrightarrow B(\widetilde{G})$ induced by f. In addition, we consider a dominant cocharacter μ of G and denote by $\widetilde{\mu}$ the dominant cocharacter of $\widetilde{G}$ associated to $f \circ \mu$. There exists a natural map*

$$\text{Gr}^{b'}_{G,\mu,b}(C) \longrightarrow \text{Gr}^{\widetilde{b}'}_{\widetilde{G},\widetilde{\mu},\widetilde{b}}(C).$$

Proof. Let us consider the natural map $\text{Gr}_{G,\mu}(C) \longrightarrow \text{Gr}_{\widetilde{G},\widetilde{\mu}}(C)$ induced by f. It suffices to show that the image of $\text{Gr}^{b'}_{G,\mu,b}(C)$ lies in $\text{Gr}^{\widetilde{b}'}_{\widetilde{G},\widetilde{\mu},\widetilde{b}}(C)$. For every G-bundle $\mathcal{E}$ on X, we write $\widetilde{\mathcal{E}}$ for the corresponding $\widetilde{G}$-bundle on X. Given an arbitrary point $x \in \text{Gr}^{b'}_{G,\mu,b}(C)$, we denote by $\widetilde{x}$ its image in $\text{Gr}_{\widetilde{G}}(C)$ and find

$$\mathcal{E}_{\widetilde{b}'} \cong \widetilde{\mathcal{E}}_{b'} \simeq \widetilde{\mathcal{E}}_{b,x} \cong \mathcal{E}_{\widetilde{b},\widetilde{x}},$$

thereby deducing that $\widetilde{x}$ lies in $\text{Gr}^{\widetilde{b}'}_{\widetilde{G},\widetilde{\mu},\widetilde{b}}(C)$ as desired. $\qquad\square$

Definition 4.16. Let us regard elements of $\mathcal{N}(G)$ as dominant rational cocharacters. We define the *Bruhat order* $\leq$ on $\mathcal{N}(G)$ by writing $\mu \leq \mu'$ if $\mu' - \mu$ is a linear combination of positive coroots with nonnegative coefficients.

Proposition 4.17. *Given a closed embedding $G \hookrightarrow \widetilde{G}$ of reductive groups over E, the induced map $\mathcal{N}(G) \longrightarrow \mathcal{N}(\widetilde{G})$ is compatible with the Bruhat order.*

Proof. The assertion is straightforward to verify by definition. $\qquad\square$

Example 4.18. Let us describe the Bruhat order for groups considered in Example 3.13.

(1) Given two elements $\mu = (\mu_1, \cdots, \mu_n)$ and $\mu' = (\mu'_1, \cdots, \mu'_n)$ of $\mathcal{N}(\mathrm{GL}_n)$, we have $\mu \leq \mu'$ if and only if the following equivalent conditions are satisfied:

  (i) We have inequalities
$$\sum_{i=1}^{j} \mu_i \leq \sum_{i=1}^{j} \mu'_i \quad \text{for } j = 1, \cdots, n$$
  with equality for $j = n$.

  (ii) If μ and μ' are regarded as polygons on the interval $[0, n]$, then μ lies below μ' with the same endpoints.

(2) Since $\mathcal{N}(\mathrm{GSp}_{2n})$ is a subset of $\mathcal{N}(\mathrm{GL}_{2n})$, Proposition 4.17 implies that the Bruhat order on $\mathcal{N}(\mathrm{GL}_n)$ restricts to the Bruhat order on $\mathcal{N}(\mathrm{GSp}_{2n})$.

Definition 4.19. Let μ be a dominant cocharacter of G.

(1) The *dual* of μ is the unique dominant cocharacter μ^* in the conjugacy class of $-\mu$.

(2) The *Galois average* of μ is defined by
$$\mu^{\Diamond} := \frac{1}{[\Gamma : \Gamma_\mu]} \sum_{\tau \in \Gamma/\Gamma_\mu} \tau(\mu) \in \mathcal{N}(G)$$
where Γ_μ denotes the stabilizer of μ in Γ.

(3) The *degree* of μ, denoted by $\mu^{\sharp}$, is the image of μ under the natural projection map $X_*(T) \twoheadrightarrow \pi_1(G)_\Gamma$.

Example 4.20. Let us illustrate Definition 4.19 for groups considered in Example 3.13.

(1) For GL_n, the Galois group Γ acts trivially on cocharacters as GL_n is split. Hence all dominant cocharacters of GL_n are elements of $\mathcal{N}(\mathrm{GL}_n)$. Given a dominant cocharacter $\mu = (\mu_1, \cdots, \mu_n)$ of GL_n, we have
$$\mu^* = (-\mu_n, \cdots, -\mu_1), \quad \mu^{\Diamond} = \mu = (\mu_1, \cdots, \mu_n), \quad \mu^{\sharp} = \mu_1 + \cdots + \mu_n.$$

(2) For GSp_{2n}, the Galois group Γ acts trivially on cocharacters as GSp_{2n} is split. Hence all dominant cocharacters of GSp_{2n} are elements of $\mathcal{N}(\mathrm{GSp}_{2n})$. Given a dominant cocharacter $\mu = (\mu_1, \cdots, \mu_{2n})$ of GSp_{2n}, we have
$$\mu^* = (-\mu_{2n}, \cdots, -\mu_1), \quad \mu^{\Diamond} = \mu = (\mu_1, \cdots, \mu_{2n}), \quad \mu^{\sharp} = \frac{1}{n}(\mu_1 + \cdots + \mu_{2n}).$$

Proposition 4.21 ([CFS21, Proposition 5.2], [Vie21, Corollary 5.4]). *Let μ be a dominant cocharacter of G. Take elements $b, b' \in B(G)$ such that b is basic. The Newton stratum $\mathrm{Gr}_{G,\mu,b}^{b'}$ is not empty if and only if we have $\kappa(b') = \kappa(b) - \mu^{\sharp}$ and $\nu(b') \leq \nu(b) + (\mu^*)^{\Diamond}$.*

5. Nonempty Newton strata in minuscule Schubert cells for GSp_{2n}

Definition 5.1. Given a rational tuple η, we denote its i-th entry by $\lambda_i(\eta)$.

(1) The *identity cocharacter* of GSp_{2n} is the element $\mathbb{1} \in \mathcal{N}(\mathrm{GSp}_{2n})$ with
$$\lambda_i(\mathbb{1}) = 1 \quad \text{for } i = 1, \cdots, 2n.$$

(2) The *ordinary cocharacter* of GSp_{2n} is the element $\mu_{\mathrm{ord}} \in \mathcal{N}(\mathrm{GSp}_{2n})$ with
$$\lambda_i(\mu_{\mathrm{ord}}) = \begin{cases} 1 & \text{for } i = 1, \cdots, n, \\ 0 & \text{for } i = n+1, \cdots, 2n. \end{cases}$$

(3) We define the *slopewise dominance order* $\preceq$ on $\mathcal{N}(\mathrm{GSp}_{2n})$ by writing $\mu \preceq \mu'$ if we have $\lambda_i(\mu) \le \lambda_i(\mu')$ for $i = 1, \cdots, 2n$.

Remark. Our terminologies for $\mathbb{1}$ and μ_{ord} come from the following facts:

(a) As a polygon, $\mathbb{1}$ coincides with the graph of the identity function on $[0, 2n]$.
(b) As a polygon, μ_{ord} coincides with the Newton polygon of the ordinary polarized abelian variety of dimension $2n$.

Proposition 5.2. *The natural map $B(\mathrm{GSp}_{2n}) \longrightarrow B(\mathrm{GL}_{2n})$ is injective.*

Proof. The assertion is evident by the commutative diagram

$$
\begin{array}{ccc}
B(\mathrm{GL}_{2n}) & \overset{\nu}{\hookrightarrow} & \mathcal{N}(\mathrm{GL}_{2n}) \\
\uparrow & & \uparrow \\
B(\mathrm{GSp}_{2n}) & \overset{\nu}{\hookrightarrow} & \mathcal{N}(\mathrm{GSp}_{2n})
\end{array}
$$

where the Newton maps are injective as noted in Example 3.13. $\qquad\square$

Remark. While Proposition 5.2 is an interesting fact, it is not essential for our argument as we will use it only for notational convenience. Every part in our argument that invokes Proposition 5.2 will rely only on the injectivity of the map $\mathcal{N}(\mathrm{GSp}_{2n}) \hookrightarrow \mathcal{N}(\mathrm{GL}_{2n})$.

Proposition 5.3 ([Hon22, Lemma 3.1.4 and Proposition 3.3.2]). *Let us regard $\mathbb{1}$ and μ_{ord} as dominant cocharacters of GL_{2n}. Consider elements $b, b' \in B(\mathrm{GL}_{2n})$.*

(1) If the Newton stratum $\mathrm{Gr}^{b'}_{\mathrm{GL}_{2n}, \mathbb{1}, b}$ contains $\widetilde{x} := \mathbb{1}(t)\,\mathrm{GL}_{2n}(B^+_{\mathrm{dR}}) \in \mathrm{Gr}_{\mathrm{GL}_{2n}, \mathbb{1}}(C)$, we have $\nu(b') = \nu(b) - \mathbb{1}$.

(2) If the Newton stratum $\mathrm{Gr}^{b'}_{\mathrm{GL}_{2n}, \mu_{\mathrm{ord}}, b}$ is not empty, we have
$$\nu(b') \le \nu(b) + \mu_{\mathrm{ord}}^* \quad \text{and} \quad \nu(b') \preceq \nu(b) \preceq \nu(b') + \mathbb{1}.$$

Remark. In fact, the second statement holds for any dominant cocharacter μ of GL_{2n} whose entries are either 0 or 1.

Lemma 5.4. *Given elements $b, b' \in B(\mathrm{GSp}_{2n})$ such that the Newton stratum $\mathrm{Gr}^{b'}_{\mathrm{GSp}_{2n}, \mathbb{1}, b}$ contains $x := \mathbb{1}(t)\,\mathrm{GSp}_{2n}(B^+_{\mathrm{dR}}) \in \mathrm{Gr}_{\mathrm{GSp}_{2n}, \mathbb{1}}(C)$, we have $\nu(b') = \nu(b) - \mathbb{1}$.*

Proof. We may regard $\mathbb{1}$ as a dominant cocharacter of GL_{2n}. In addition, in light of Proposition 5.2, we may regard b and b' as elemnts of $B(\mathrm{GL}_{2n})$. By Proposition 4.15, we have a natural map
$$\mathrm{Gr}^{b'}_{\mathrm{GSp}_{2n}, \mathbb{1}, b}(C) \longrightarrow \mathrm{Gr}^{b'}_{\mathrm{GL}_{2n}, \mathbb{1}, b}(C)$$
which sends x to $\widetilde{x} := \mathbb{1}(t)\,\mathrm{GL}_{2n}(B^+_{\mathrm{dR}})$. Hence the desired assertion is evident by Proposition 5.3. $\qquad\square$

Proposition 5.5. *Let μ be a dominant cocharacter of GSp_{2n} with nonnegative entries. Given arbitrary elements $b, b' \in B(\mathrm{GSp}_{2n})$, we have the following equivalent conditions:*

(i) $\mathrm{Gr}_{\mathrm{GSp}_{2n}, \mu, b}^{b'}$ is nonempty.

(ii) $\mathrm{Gr}_{\mathrm{GSp}_{2n}, \mu^*, b'}^{b}$ is nonempty.

(iii) $\mathrm{Gr}_{\mathrm{GSp}_{2n}, \mu+\mathbb{1}, b}^{\widetilde{b'}}$ is nonempty for $\widetilde{b'} \in B(\mathrm{GSp}_{2n})$ with $\nu(\widetilde{b'}) = \nu(b') - \mathbb{1}$.

Proof. For an arbitrary point $x = g\mu(t)\,\mathrm{GSp}_{2n}(B_{\mathrm{dR}}^+) \in \mathrm{Gr}_{\mathrm{GSp}_{2n}, \mu, b}^{b'}(C)$, we take

$$x^* := g^{-1}\mu^*(t)\,\mathrm{GSp}_{2n}(B_{\mathrm{dR}}^+) \in \mathrm{Gr}_{\mathrm{GSp}_{2n}, \mu^*}(C)$$

and find $\mathcal{E}_{b', x^*} \simeq \mathcal{E}_b$, thereby deducing that x^* lies in $\mathrm{Gr}_{\mathrm{GSp}_{2n}, \mu^*, b'}^{b}(C)$. Similarly, every point in $\mathrm{Gr}_{\mathrm{GSp}_{2n}, \mu^*, b'}^{b}(C)$ yields a point in $\mathrm{Gr}_{\mathrm{GSp}_{2n}, \mu, b}^{b'}(C)$. Hence we establish the equivalence of the conditions (i) and (ii) by Proposition 4.14.

It remains to verify the equivalence of the conditions (i) and (iii). For an arbitrary point $x = g\mu(t)\,\mathrm{GSp}_{2n}(B_{\mathrm{dR}}^+) \in \mathrm{Gr}_{\mathrm{GSp}_{2n}, \mu, b}^{b'}(C)$, we take

$$\widetilde{x} := g\mu(t)\mathbb{1}(t)\,\mathrm{GSp}_{2n}(B_{\mathrm{dR}}^+) \in \mathrm{Gr}_{\mathrm{GSp}_{2n}, \mu+\mathbb{1}}(C)$$

and find $\mathcal{E}_{b, \widetilde{x}} \simeq \mathcal{E}_{\widetilde{b'}}$ by Lemma 5.4, thereby deducing that $\widetilde{x}$ lies in $\mathrm{Gr}_{\mathrm{GSp}_{2n}, \mu+\mathbb{1}, b}^{\widetilde{b'}}(C)$. Similarly, every point in $\mathrm{Gr}_{\mathrm{GSp}_{2n}, \mu+\mathbb{1}}(C)$ gives rise to a point in $\mathrm{Gr}_{\mathrm{GSp}_{2n}, \mu, b}^{b'}(C)$. Hence we complete the proof by Proposition 4.14. $\qquad\square$

Proposition 5.6. *For a minuscule dominant cocharacter μ of GSp_{2n}, we have $\mu = d \cdot \mathbb{1}$ or $\mu = d \cdot \mathbb{1} + \mu_{\mathrm{ord}}$ for some $d \in \mathbb{Z}$.*

Proof. Let us identify the character lattice of GL_{2n} with $\mathbb{Z}^{2n}$ and write $e_1, \cdots, e_{2n}$ for its standard basis elements. Then we can identify the character lattice of GSp_{2n} as the lattice generated by

$$e_0' := e_{n+1} + \cdots + e_{2n} \quad \text{and} \quad e_i' := e_i - e_{2n+1-i} \text{ for } i = 1, \cdots, n.$$

The set of positive roots for GL_{2n} and GSp_{2n} are given by

$$\Phi_{\mathrm{GL}_{2n}}^+ = \{e_i - e_j : 1 \le i < j \le 2n\},$$
$$\Phi_{\mathrm{GSp}_{2n}}^+ = \{e_i' - e_j' : 1 \le i < j \le n\} \cup \{e_i' + e_j' - e_0' : 1 \le i \le j \le n\}.$$

Now the assertion is straightforward to verify. $\qquad\square$

Proposition 5.7. *Let μ be a minuscule dominant cocharacter of GSp_{2n}. Assume that all entries of μ are either d or $d+1$ for some $d \in \mathbb{Z}$. For elements $b, b' \in B(\mathrm{GSp}_{2n})$ such that the Newton stratum $\mathrm{Gr}_{\mathrm{GSp}_{2n}, \mu, b}^{b'}$ is not empty, we have*

$$\nu(b') \le \nu(b) + \mu^* \quad \text{and} \quad \nu(b') + d \cdot \mathbb{1} \preceq \nu(b) \preceq \nu(b') + (d+1) \cdot \mathbb{1}.$$

Proof. By Propositions 5.5 and 5.6, we may take $d = 0$ and assume that μ is either 0 or μ_{ord}. For $\mu = 0$, the assertion is evident since the only nonempty Newton stratum in $\mathrm{Gr}_{\mathrm{GSp}_{2n}, 0}$ with respect to b is $\mathrm{Gr}_{\mathrm{GSp}_{2n}, 0, b}^{b}$. For $\mu = \mu_{\mathrm{ord}}$, we have a natural map

$$\mathrm{Gr}_{\mathrm{GSp}_{2n}, \mu_{\mathrm{ord}}, b}^{b'}(C) \longrightarrow \mathrm{Gr}_{\mathrm{GL}_{2n}, \mu_{\mathrm{ord}}, b}^{b'}(C)$$

given by Proposition 4.15, where for $\mathrm{Gr}_{\mathrm{GL}_{2n}, \mu_{\mathrm{ord}}, b}^{b'}$ we regard μ_{ord} as a dominant cocharacter of GL_{2n} and b, b' as elements of $B(\mathrm{GL}_{2n})$ in light of Proposition 5.2, and thus deduce the desired assertion by Propositions 4.14 and 5.3. $\qquad\square$

Definition 5.8. Let $\alpha = (\alpha_1, \cdots, \alpha_l)$ be an ordered partition of an integer $m \leq n$.

(1) We define the associated Levi subgroup M_α of GSp_{2n} to be the group of block diagonal matrices

$$\begin{pmatrix} g_1 & & & & & & \\ & \ddots & & & & & \\ & & g_l & & & & \\ & & & h & & & \\ & & & & \psi(h)(g_l^T)^{-1} & & \\ & & & & & \ddots & \\ & & & & & & \psi(h)(g_1^T)^{-1} \end{pmatrix} \quad \text{with } g_i \in \mathrm{GL}_{\alpha_i}, h \in \mathrm{GSp}_{2(n-m)},$$

where ψ denotes the similitude character of $\mathrm{GSp}_{2(n-m)}$.

(2) Given an element $b \in B(\mathrm{GSp}_{2n})$, its *reduction to M_α* is an element $\bar{b} \in B(M_\alpha)$ whose image under the natural map $B(M_\alpha) \longrightarrow B(\mathrm{GSp}_{2n})$ is equal to b.

Remark. It is not hard to show that every standard Levi subgroup of GSp_{2n} is equal to M_α for some ordered partition α of an integer $m \leq n$.

Proposition 5.9. *Let $\alpha = (\alpha_1, \cdots, \alpha_l)$ be an ordered partition of an integer $m \leq n$. We write $n_i := \alpha_1 + \cdots + \alpha_i$ for $i = 1, \cdots, l$. In addition, we set $n_0 := 0$ and $n_{l+1} := n$.*

(1) *The Newton set of M_α is the set of dominant rational cocharacters of M_α, which is also identified with the set of tuples $(\lambda_1, \cdots, \lambda_{2n}) \in \mathbb{Q}^{2n}$ satisfying the following properties:*

   (i) *We have $\lambda_{n_{i-1}+1} \geq \cdots \geq \lambda_{n_i}$ for $i = 1, \cdots, l+1$.*

   (ii) *There exists a rational number d with $\lambda_i + \lambda_{2n+1-i} = d$ for $i = 1, \cdots, n$.*

(2) *Given two elements $\mu = (\mu_1, \cdots, \mu_{2n})$ and $\mu' = (\mu_1', \cdots, \mu_{2n}')$ of $\mathcal{N}(M_\alpha)$, the inequality $\mu \leq \mu'$ holds if and only if we have inequalities*

$$\sum_{i=1}^{j} \mu_i \leq \sum_{i=1}^{j} \mu_i' \quad \text{for } j = 1, \cdots, 2n$$

   with equalities for $j = n_1, \cdots, n_{l+1}$.

(3) *There exists a natural isomorphism $\pi_1(M_\alpha)_\Gamma \cong \mathbb{Z}^{l+1}$.*

(4) *Given a dominant cocharacter $\mu = (\mu_1, \cdots, \mu_{2n})$ of M_α, its dual $\mu^* = (\mu_1^*, \cdots, \mu_{2n}^*)$ is specified by the following identities:*

   • $\mu_{n_{i-1}+j}^* = -\mu_{n_i+1-j}$ *for $i = 1, \cdots, l$ and $j = 1, \cdots, \alpha_i$.*

   • $\mu_{m+i}^* = -\mu_{2n+1-m-i}$ *for $i = 1, \cdots, n-m$.*

   • $\mu_i^* + \mu_{2n+1-i}^* = -(\mu_i + \mu_{2n+1-i})$ *for $i = 1, \cdots, n$.*

(5) *Given a dominant cocharacter $\mu = (\mu_1, \cdots, \mu_{2n})$ of M_α, its degree $\mu^\sharp = (\mu_1^\sharp, \cdots, \mu_{l+1}^\sharp)$ is given by*

$$\mu_i^\sharp = \begin{cases} \mu_{n_{i-1}+1} + \cdots + \mu_{n_i} & \text{for } i = 1, \cdots, l \\[2ex] \dfrac{\mu_{m+1} + \cdots \mu_{2n-m}}{n-m} & \text{for } i = l+1 \end{cases}.$$

Proof. All statements are straightforward to verify by our discussion in Examples 3.13, 4.18, and 4.20. $\qquad\square$

Proposition 5.10. *Let $\alpha = (\alpha_1, \cdots, \alpha_l)$ be an ordered partition of an integer $m \leq n$. We write $n_i = \alpha_1 + \cdots + \alpha_i$ for $i = 1, \cdots, l$. In addition, we set $n_0 := 0$ and $n_{l+1} := n$. Given an element $b \in B(\mathrm{GSp}_{2n})$ such that $\nu(b)$ has integer points with $n_1, \cdots, n_l$ as x-coordinates, there exists a reduction of b to M_α.*

Proof. For a rational tuple η, we denote its i-th entry by $\lambda_i(\eta)$. Our discussion in Example 3.13 implies that there exists an integer d with

$$\lambda_i(\nu(b)) + \lambda_{2n+1-i}(\nu(b)) = d \quad \text{for } i = 1, \cdots, n. \tag{5.1}$$

For each $i = 1, \cdots, l$, we define α_i-tuples $\nu^{(i)}$ and $\widehat{\nu}^{(i)}$ by

$$\lambda_j(\nu^{(i)}) = \lambda_{n_{i-1}+j}(\nu(b)) \quad \text{and} \quad \lambda_j(\widehat{\nu}^{(i)}) = \lambda_{2n-n_i+j}(\nu(b)) \quad \text{for } j = 1, \cdots, \alpha_j.$$

In addition, we define a $2(n-m)$-tuple ω by

$$\lambda_i(\omega) = \lambda_{m+i}(\nu(b)) \quad \text{for } i = 1, \cdots, 2(n-m).$$

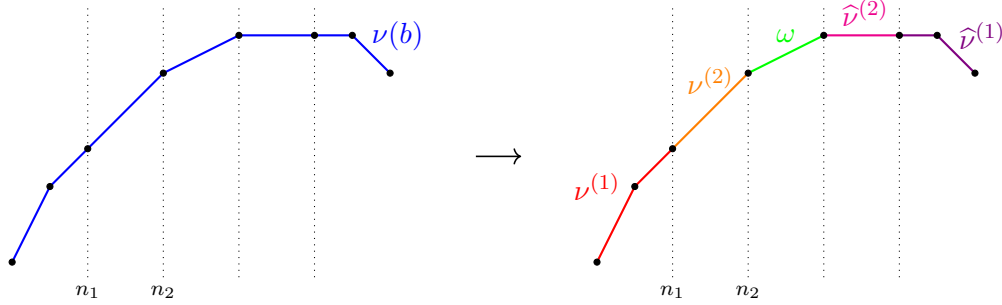As polygons, these tuples form a partition of $\nu(b)$ as illustrated in Figure 2.

FIGURE 2. Partition of $\nu(b)$ by integer points

By construction, these polygons all have integer breakpoints. In addition, by (5.1) we find

$$\lambda_i(\omega) + \lambda_{2n-2m+1-i}(\omega) = d \quad \text{for } i = 1, \cdots, 2(n-m). \tag{5.2}$$

Hence by our discussion in Example 3.13, we find an element $b_i \in B(\mathrm{GL}_{\alpha_i})$ with $\nu(b_i) = \nu^{(i)}$ for $i = 1, \cdots, l$ and also an element $c \in B(\mathrm{GSp}_{2(n-m)})$ with $\nu(c) = \omega$. Let us choose a representative $g_i \in \mathrm{GL}_{\alpha_i}(\breve{E})$ of b_i for each $i = 1, \cdots, l$ and a representative $h \in \mathrm{GSp}_{2(n-m)}(\breve{E})$ of c. We take $\bar{b} \in B(M_\alpha)$ to be the σ-conjugacy class of

$$\begin{pmatrix} g_1 & & & & & & \\ & \ddots & & & & & \\ & & g_l & & & & \\ & & & h & & & \\ & & & & \psi(h)(g_l^T)^{-1} & & \\ & & & & & \ddots & \\ & & & & & & \psi(h)(g_1^T)^{-1} \end{pmatrix} \in M_\alpha(\breve{E})$$

where ψ denotes the similitude character of $\mathrm{GSp}_{2(n-m)}$, and denote by $\dot{b}$ the image of $\bar{b}$ under the natural map $B(M_\alpha) \longrightarrow B(\mathrm{GSp}_{2n})$. By the functoriality of the Newton map, $\nu(\dot{b})$ is given by the image of $\nu(\bar{b})$ under the natural map $\mathcal{N}(M_\alpha) \longrightarrow \mathcal{N}(\mathrm{GSp}_{2n})$. It is then not hard to see by (5.1) and (5.2) that $\nu(\dot{b})$ coincides with $\nu(b)$. Since the Newton map for GSp_{2n} is injective as noted in Example 3.13, we find $b = \dot{b}$ and consequently complete the proof. $\square$

Theorem 5.11. *Let μ be a minuscule dominant cocharacter of GSp_{2n} whose entries are either d or $d+1$ for some $d \in \mathbb{Z}$. Consider elements $b, b' \in B(\mathrm{GSp}_{2n})$ such that any two distinct slopes in $\nu(b)$ differ by more than 1. The Newton stratum $\mathrm{Gr}^{b'}_{\mathrm{GSp}_{2n}, \mu, b}$ is nonempty if and only if the following conditions are satisfied:*

   (i) *We have $\nu(b') \leq \nu(b) + \mu^*$ and $\nu(b') + d \cdot \mathbb{1} \preceq \nu(b) \preceq \nu(b') + (d+1) \cdot \mathbb{1}$.*

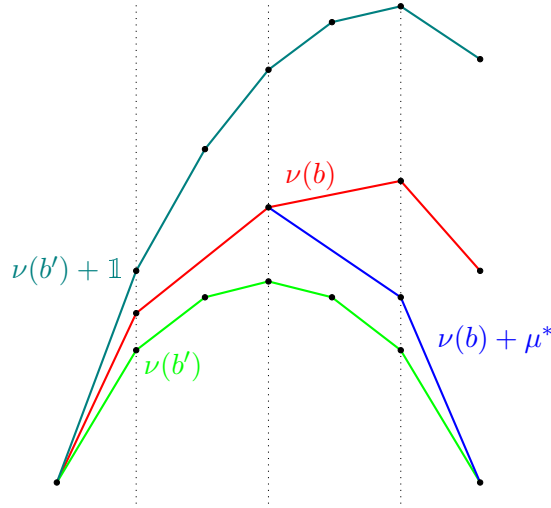   (ii) *For each breakpoint of $\nu(b)$, there exists a breakpoint of $\nu(b')$ with the same x-coordinate.*

FIGURE 3. Illustration of the conditions in Theorem 5.11 for $d = 0$

Proof. For a rational tuple η, we denote its i-th entry by $\lambda_i(\eta)$. By Propositions 5.5 and 5.6, we may set $d = 0$ and assume that μ is either 0 or μ_{ord}. For $\mu = 0$, the assertion is evident since the only nonempty Newton stratum in $\mathrm{Gr}_{\mathrm{GSp}_{2n}, 0}$ with respect to b is $\mathrm{Gr}^b_{\mathrm{GSp}_{2n}, 0, b}$. Therefore we may henceforth take $\mu = \mu_{\mathrm{ord}}$.

Let us first assume that $\mathrm{Gr}^{b'}_{\mathrm{GSp}_{2n}, \mu, b}$ is nonempty. The necessity of the condition (i) immediately follows from Proposition 5.7. Now we take an arbitrary breakpoint of $\nu(b)$ and denote its x-coordinate by m. By the condition (i) and the assumption on slopes of $\nu(b)$, we find

$$\lambda_{m+1}(\nu(b')) \leq \lambda_{m+1}(\nu(b)) < \lambda_m(\nu(b)) - 1 \leq \lambda_m(\nu(b'))$$

and consequently deduce that μ' has a breakpoint with x-coordinate m, thereby establishing the necessity of the condition (ii).

For the converse, we now assume that the conditions (i) and (ii) are satisfied. We consider the breakpoints of $\nu(b)$ on the interval $[0, n]$ and denote their x-coordinates by $n_1, \cdots, n_l$ in ascending order. We also set $n_0 := 0$ and take $\alpha = (\alpha_1, \cdots, \alpha_l)$ to be an ordered partition of $m = n_l$ with

$$\alpha_i := n_i - n_{i-1} \quad \text{for } i = 1, \cdots, l.$$

It is not hard to see that the centralizer of $\nu(b)$ is M_α. Moreover, Proposition 5.10 and the condition (ii) together imply that b and b' admit reductions to M_α, which we respectively denote by $\bar{b}$ and $\bar{b'}$. Let us now set

$$d_i := \sum_{j=n_{i-1}+1}^{n_i} \lambda_j(\nu(b)) - \sum_{j=n_{i-1}+1}^{n_i} \lambda_j(\nu(b')) \quad \text{for } i = 1, \cdots, l.$$

By the condition (i), we have $0 \leq d_i \leq \alpha_i$ for $i = 1, \cdots, l$. Take $\overline{\mu}$ to be the $2n$-tuple specified by the following properties:

(a) For $i = 1, \cdots, l+1$ and $j = 1, \cdots, \alpha_i$, we have
$$\lambda_{n_{i-1}+j}(\overline{\mu}) = \begin{cases} 1 & \text{if } j \leq d_i, \\ 0 & \text{if } j > d_i, \end{cases}$$
where we set $d_{l+1} := n - n_l = n - m$.

(b) For $j = 1, \cdots, n$, we have $\lambda_j(\overline{\mu}) + \lambda_{2n+1-j}(\overline{\mu}) = 1$.

It is not hard to see by Proposition 5.9 that $\overline{\mu}$ is a dominant cocharacter of M_α with
$$\kappa(\overline{b}') = \kappa(\overline{b}) - \overline{\mu}^\sharp \quad \text{and} \quad \nu(\overline{b}') \leq \nu(\overline{b}) - (\overline{\mu}^*)^\Diamond.$$

Moreover, $\overline{b}$ is basic in $B(M_\alpha)$ by construction. Hence Proposition 4.21 implies that the Newton stratum $\mathrm{Gr}_{M_\alpha, \overline{\mu}, \overline{b}}^{\overline{b}'}$ is not empty. Since we have a natural map
$$\mathrm{Gr}_{M_\alpha, \overline{\mu}, \overline{b}}^{\overline{b}'}(C) \longrightarrow \mathrm{Gr}_{\mathrm{GSp}_{2n}, \mu, b}^{b'}(C)$$
by Proposition 4.15, we deduce by Proposition 4.14 that $\mathrm{Gr}_{\mathrm{GSp}_{2n}, \mu, b}^{b'}$ is not empty as desired, thereby completing the proof. $\qquad\square$

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